The Kardar-Parisi-Zhang equation and universality class II

Some exactly solvable models in the KPZ universality class

More interesting mappings

Relation to random matrices

Replica approach

Recapitulation

Non-linear growth as a *stochastic, off-equilibrium* phenomenon

with

- Self-similarity / scale invariance
- Non-trivial critical exponents
- Universality
- Exactly solvable models:
  exact exponents AND exact scaling functions!

Recapitulation

Interesting mappings: KPZ growth equation

$$\frac{\partial}{\partial t}h(\boldsymbol{x},t) = v_0 + \nu\boldsymbol{\nabla}^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla}h)^2 + \eta(\boldsymbol{x},t).$$

$\longleftrightarrow$

Burger's equation (fluid dynamics)

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v} - \lambda\nabla\eta$$

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$\longleftrightarrow$

Stochastic heat equation

$$\partial_t Z(x,t) = \nu \partial_x^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t) \eta(x,t)$$

Recapitulation

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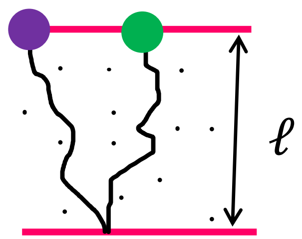
Stochastic heat equation

$$\partial_t Z(x,t) = \nu \partial_x^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t) \eta(x,t)$$

$\leftrightarrow$

Directed polymer in random media

$$\begin{aligned} Z(x,t) &= \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[-\frac{1}{T}\left(\frac{1}{2}\int_0^t \dot{u}^2 dt - \int_0^t \lambda \eta(u(t),t)\right)\right] \\ &\equiv \int_{(u(0)=0)}^{(u(t)=x)} \mathcal{D}[u(t)] \exp\left[-\frac{1}{T}\left(E_{\mathrm{el}} + E_{\mathrm{pot}}\right)\right] \end{aligned}$$



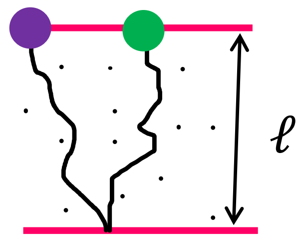
Recapitulation

Scaling of directed polymers in 1+1 dimension

Free energy fluctuations $\delta F \sim \ell^{\theta=1/3}$

Lateral displacements: “roughness” $x \sim \ell^{\zeta=2/3}$

Statistical tilt symmetry: $\theta = 2\zeta - 1$



Both reflect the dominance of disorder!

Recapitulation

Scaling of directed polymers in 1+1 dimension

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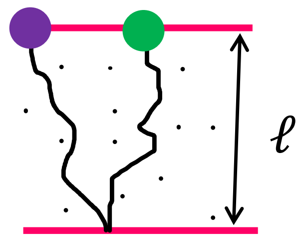


Lateral displacements: “roughness” $x \sim \ell^{\zeta=2/3}$

Statistical tilt symmetry: $\theta = 2\zeta - 1$

Notable aspects:

1. Rougher than a random walk. $\zeta > 1/2$

   Disorder dominates entropy!

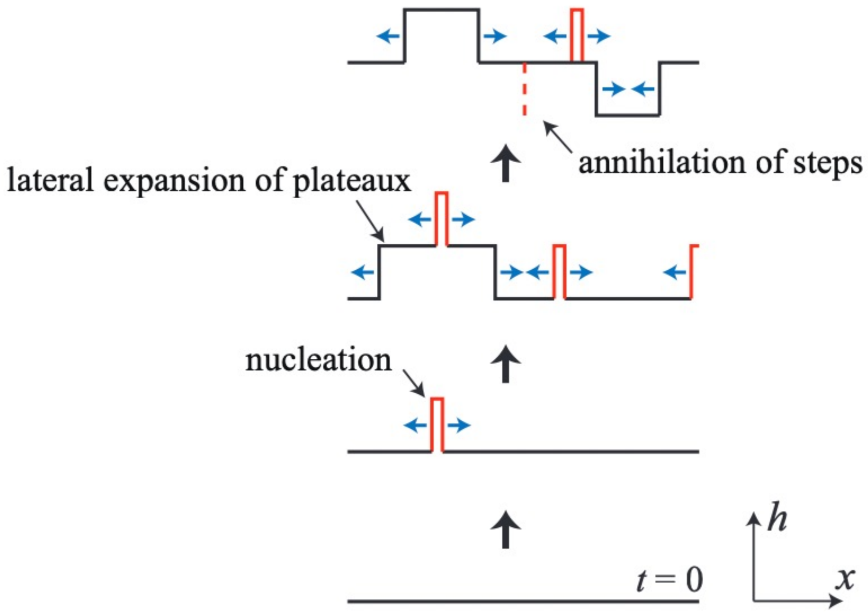
2. Free energy fluctuates less than a random sum! $\theta < 1/2$

Exactly solvable models and a few more interesting mappings

Polynuclear growth model

Discrete height : $h \in \mathbf{N}$

Continuum in space *x* and time *t*



3. Plateaux merge on meeting

2. Nuclei spread with speed *v* = *1* to both sides

1. Random nucleation events (x_n,t_n), with density $\rho\, dx dt \quad (\rho = 1)$

$$h(x_n, t_n) \to h(x_n, t_n) + 1$$

Polynuclear growth model

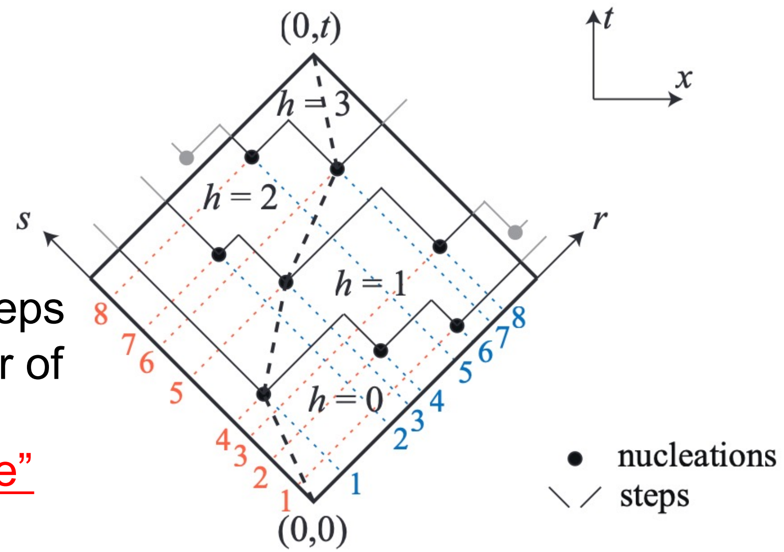
“Circular model” $\langle h(x,t)\rangle = \sqrt{2(t^2 - x^2)}$

Nucleation only in the regime $|x| \leq t$

Height profile

of nuclei N_{nuc} = Poiss(t^2/2)

h(0,t) = #height steps = maximal number of nuclei on forward pointing “world line”



Like DPRM with point defects!

Polynuclear growth model

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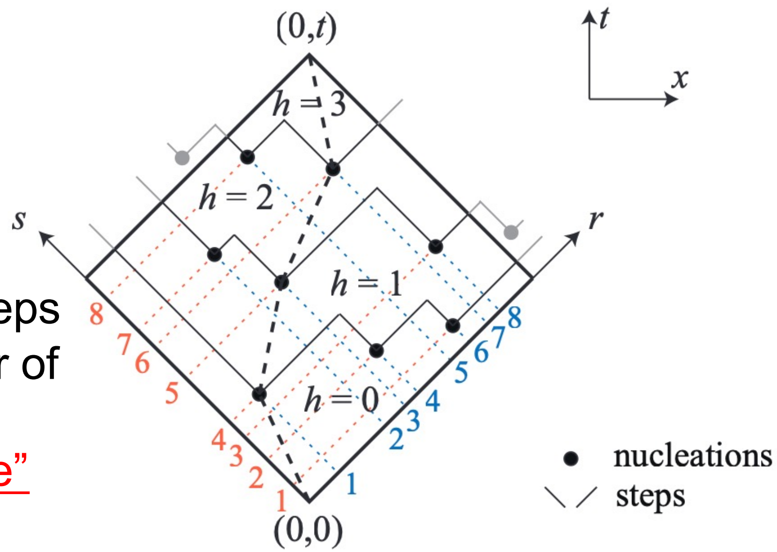

Nuclei define a random permutation!

Ulam's problem:

Maximal increasing subsequence of the permutation?

Maximizing world line ↔ length of maximal increasing subsequence = h(0,t)!

Polynuclear growth model

"Circular model" $\longleftarrow$ $\langle h(x,t)\rangle = \sqrt{2(t^2 - x^2)}$

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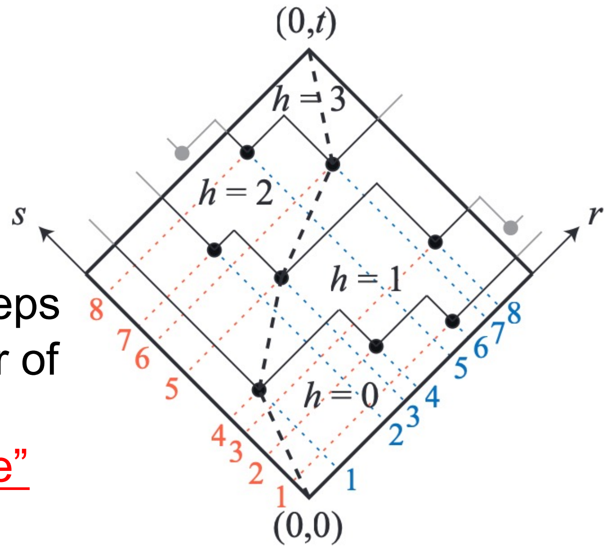


Exact solution via combinatorics!
(Baik, Deift Johansson '99)

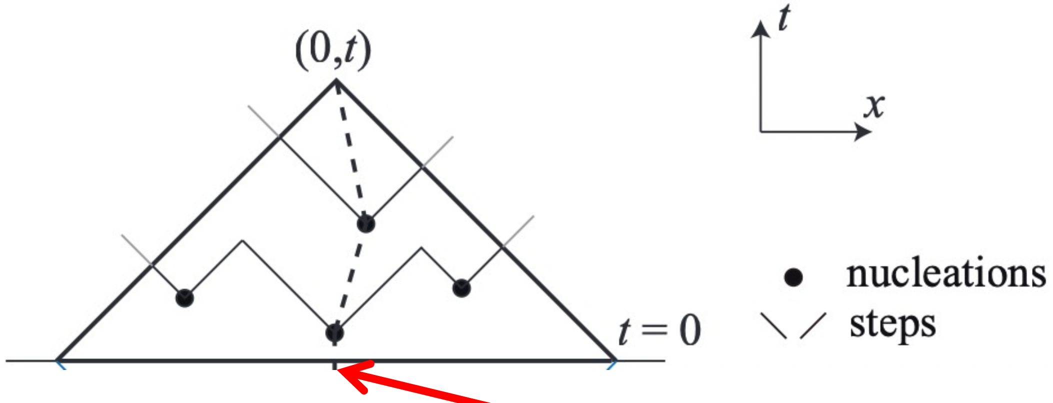
$$h(0,t) = \sqrt{2}t + (t/\sqrt{2})^{1/3}\chi_{\mathrm{TW},2}$$

Height fluctuations/length of maximal increasing subsequence are distributed according to Tracy-Widom distribution, like λ_{max} of random (Gaussian unitary GUE) matrix! WHY??

Polynuclear growth model

"Flat model": no restriction on x

Height profile



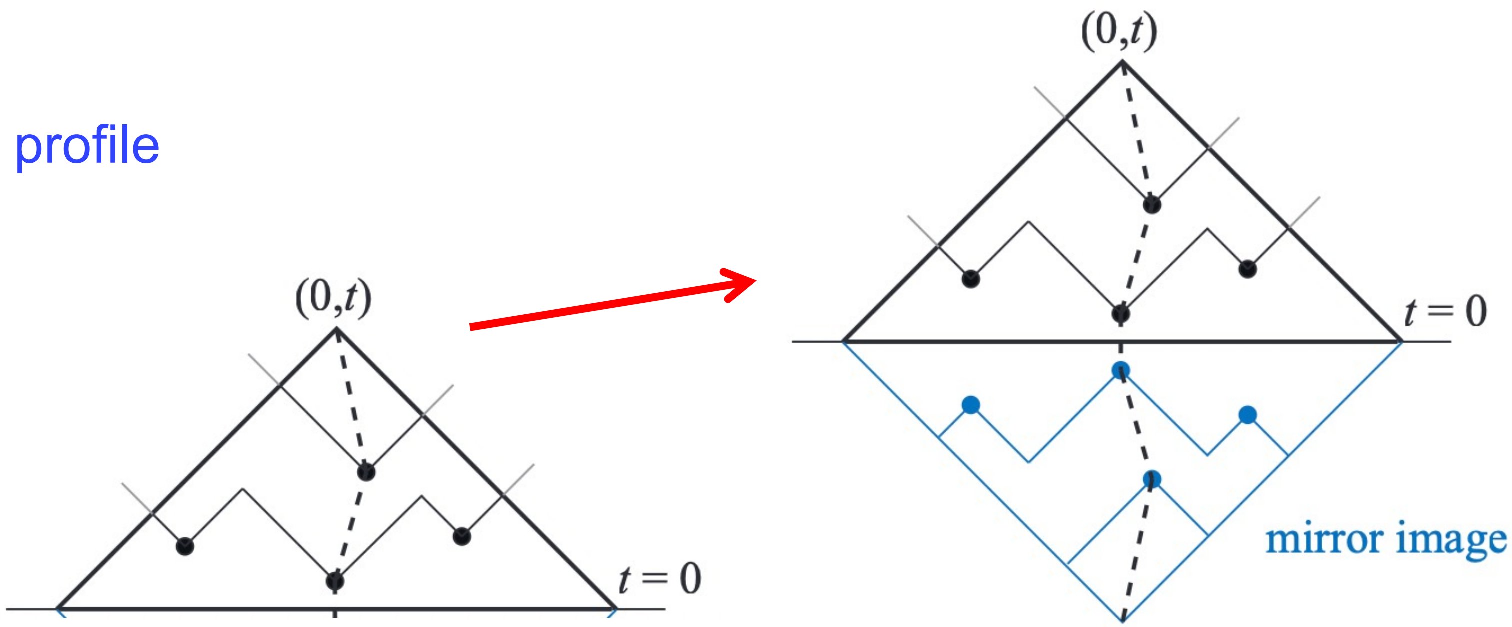
Note: Only nucleations in backward light cone of (x,t) influence h(x,t)

Path maximizing number of nuclei may start at any x at t=0 (line-to-point).

Polynuclear growth model

“Flat model”: no restriction on x

Height profile

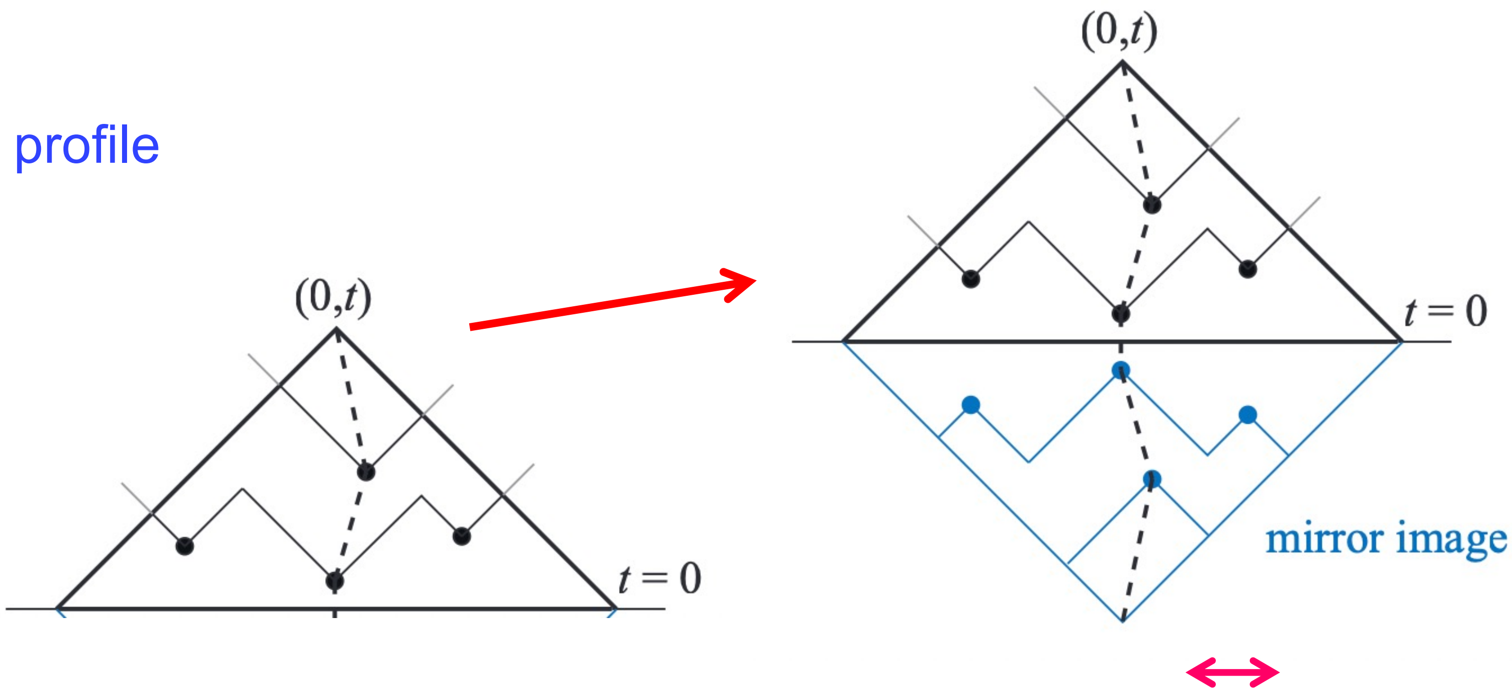


Path maximizing number of nuclei may start at any x at t=0 (line-to-point). Point-to-point from time-reversed mirror image!

Polynuclear growth model

“Flat model”: no restriction on x

Height profile



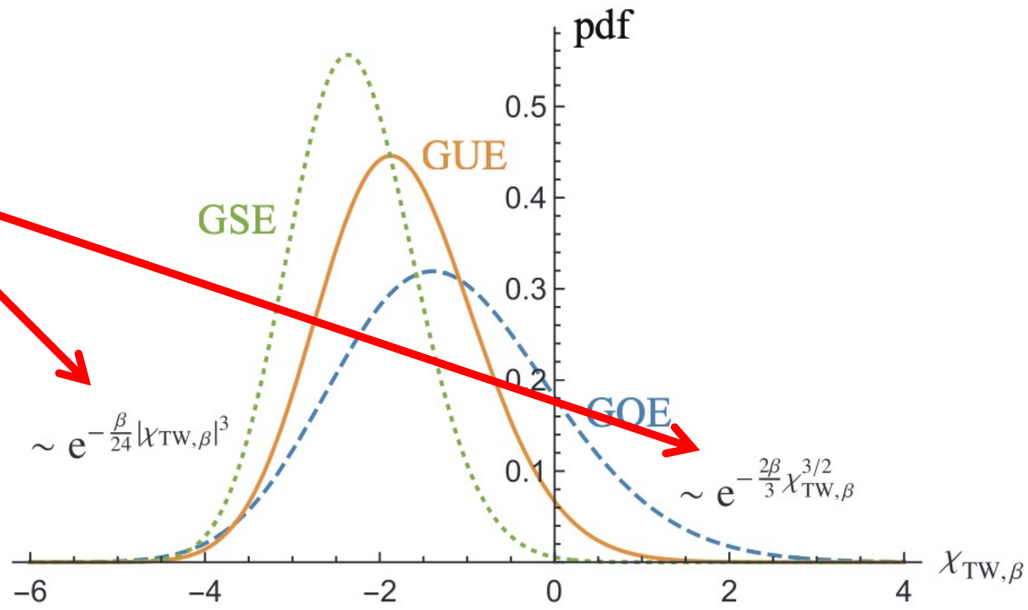
Path maximizing number of nuclei may start at any x at t=0 (line-to-point). Point-to-point from time-reversed mirror image!

$\delta h(0,t)$ distributed like λ_{max} of TRS matrix (GOE)!

Tracy-Widom distribution

Stretched exponential decays, asymmetric!

Negative height fluctuations are much more unlikely than positive ones!
(as it requires "compression" of the height profile)



Circular model: $h(0,t) = \sqrt{2}t + (t/\sqrt{2})^{1/3}\chi_{\mathrm{TW}}$ (GUE)

Flat surface: $2h(0,t) \simeq \sqrt{2}(2t) + (2t/\sqrt{2})^{1/3}\chi_{\mathrm{TW}}$ (GOE)

Exactly solvable growth model

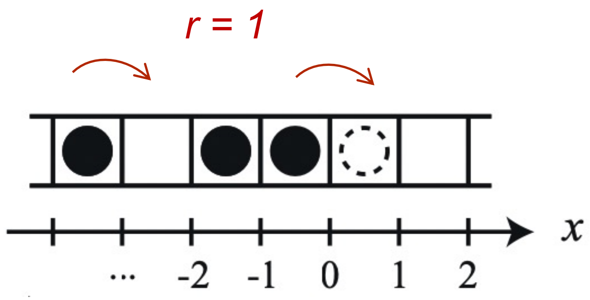
what about transport/hydrodynamics?

Exactly solvable non-linear transport processes

A model in the spirit of Burgers equation (fluid dynamics)

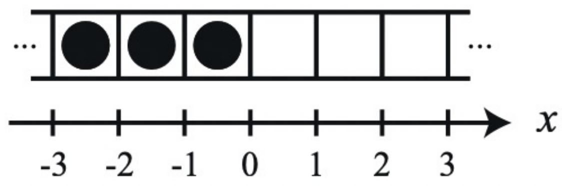
But: discrete in space, with an *exact* solution

Totally asymmetric exclusion process



Hardcore particles jump to right at rate $r = 1$ if place is empty

Initially, e.g.:



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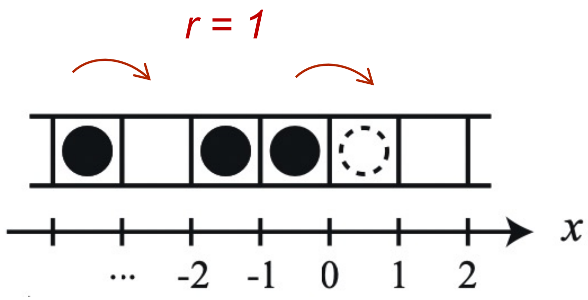

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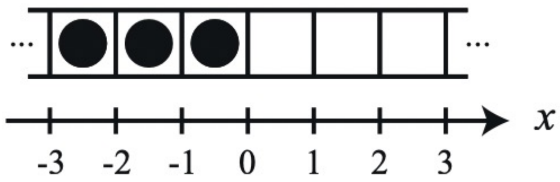
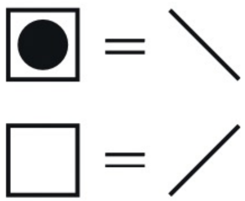
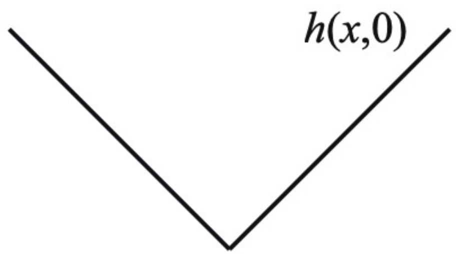


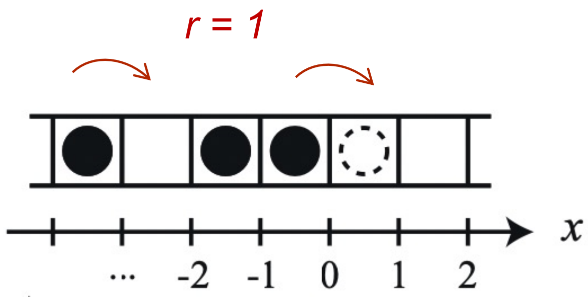
Relation to growth models:





h(0,t) = # of particles that have passed 0 until time t (integrated current).

Totally asymmetric exclusion process



Map to a directed polymer: 3 2 1 Particle No.

Define $t_{i,j} :=$ time, when particle i makes jump j

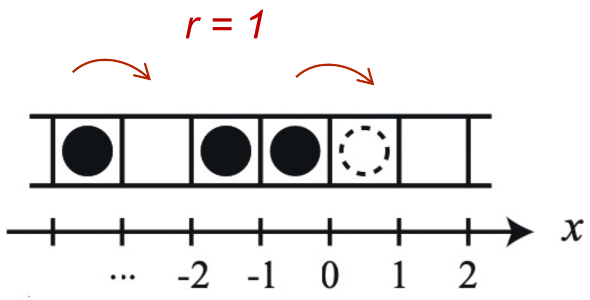
$\tau_{i;j} :=$ time for particle i to make jump j once the next site is free

$\longrightarrow$ Random variable: $p(\tau_{i;j}) = e^{-\tau_{i;j}}$

Two conditions for jump: i) particle *i-1* has jumped *j* times; ii) particle *i* has jumped *j-1* times

$$\longrightarrow \quad t_{i,j} = \max(t_{i,j-1}, t_{i-1,j}) + \tau_{i;j}$$

Totally asymmetric exclusion process



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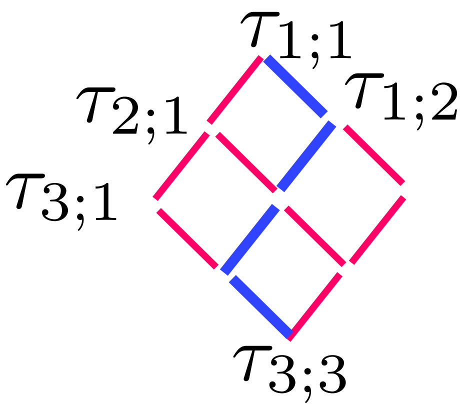

$$\longrightarrow \quad t_{i,j} = \max(t_{i,j-1}, t_{i-1,j}) + \tau_{i;j}$$

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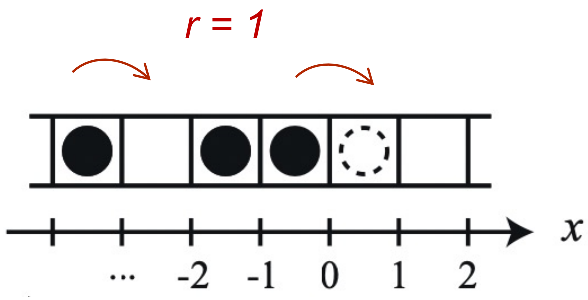

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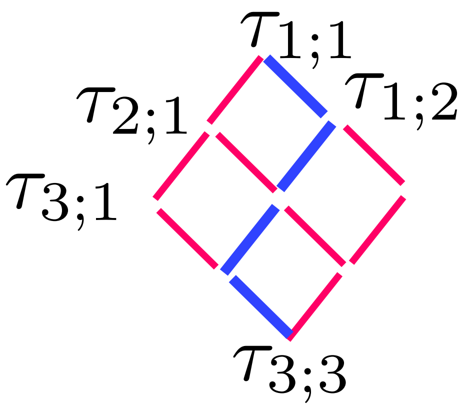

Time T_N when the N'th particle has crossed 0:

$$T_N \equiv t_{N,N} = \max_{dir.\,paths\,p} \left(\sum_{(i;j)\in p} \tau_{i;j} \right)$$

$\leftrightarrow$ minimizing the random "energy" $(-\sum \tau)$!

Special exactly solvable DPRM problem on a lattice!

$$\longrightarrow \quad t_{i,j} = \max(t_{i,j-1}, t_{i-1,j}) + \tau_{i;j}$$

Totally asymmetric exclusion process

Exact solution by Johansson (2010)

$$\mathrm{Prob}[N(t) \geq N] = \mathrm{Prob}[T_N \leq t] \propto \int_{[0,t]^N} \prod_{i=1}^{N} \mathrm{d}x_i \prod_{i<j} (x_i - x_j)^2 \prod_i \mathrm{e}^{-x_i}.$$

N(t) : = total current through 0 until time t = h(0,t) in growth picture

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Amazingly similar to distribution for largest eigenvalue of a GUE random matrix (cf later lectures):

$$\mathrm{Prob}[\lambda_{\max} \leq x] = \mathrm{Prob}[\lambda_1, \cdots, \lambda_N \leq x] = \frac{1}{Z} \int_{(-\infty,x]^N} \prod_i \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_i \mathrm{e}^{-\frac{\beta}{2}\lambda_i^2} \qquad \beta = 2$$

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Difference turns out irrelevant

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$\longleftrightarrow$ Height 

$$h(0,t) = N(t) \simeq \frac{t}{4} - 2^{-4/3} t^{1/3} \chi_2.$$

Again:
Tracy-Widom distribution!

Testing universality of KPZ

Amazing achievement:

Specific discrete models: off-equilibrium, random processes that have exact solutions!

Like Onsager solution for 2d Ising model – but off equilibrium and stochastic!

Several exact solutions:

- Polynuclear growth model
- Asymmetric exclusion processes
- Replica approach to directed polymers (see below)

All predict
- Height distribution $P\left(\frac{h(x,t)-tv_0}{t^{1/3}}\right)$: Tracy Widom distribution!
- Two point correlation function $\langle \delta h(x,t)\delta h(x',t')\rangle$ (explicit but complicated)

→ Strong indication of universality. But: does it also hold for non-solvable systems?

Calibrating KPZ parameters

In experiment (or numerics): Determine parameters A and λ :

$$A = \frac{D}{2\nu}$$ from measuring heights at constant time:

$$\langle [h(x+\ell,t) - h(x,t)]^2 \rangle \simeq A\ell,$$

Calibrating KPZ parameters

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$$\langle [h(x+\ell,t) - h(x,t)]^2 \rangle \simeq A\ell,$$

λ : via tilt symmetry

$$h_{\mathrm{new}}(\mathbf{x},t) = h(\mathbf{x}+\lambda\mathbf{s}t,t) + \mathbf{s}\cdot\mathbf{x} + \frac{\lambda}{2}\mathbf{s}^2 t$$

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$$s = \lim_{t\to\infty} \left\langle \frac{\partial h}{\partial x} \right\rangle \qquad v_\infty = v_\infty(s) = \lim_{t\to\infty} \left\langle \frac{\partial h}{\partial t} \right\rangle$$

Calibrating KPZ parameters

Finding the appropriate dimensionless coordinates to compare universal feature?

Calibrating KPZ parameters

In experiment (or numerics): Determine parameters A and λ :

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$$\boxed{\lambda = \left.\frac{d^2 v_\infty(s)}{ds^2}\right|_{s=0}}$$

Calibrating KPZ parameters

Make x and h dimensionless using A, λ, t

Calibrating KPZ parameters

Make *x* and *h* dimensionless using *A*, λ, *t*

Define $\Gamma \equiv \frac{1}{2} A^2 \lambda,$

Crossover spatial scale at time t: $\xi(t) = 2(\Gamma t)^{2/3}/A$

Rescaled space coordinate $\zeta \equiv \frac{x}{\xi(t)}$

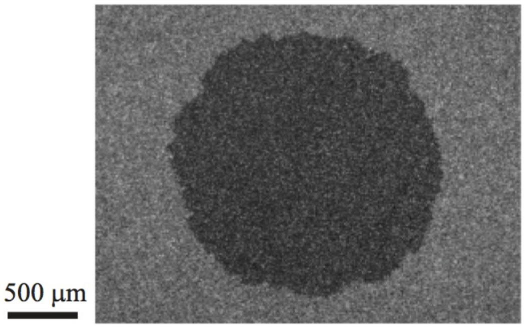
Rescaled height fluctuation $h(x,t) \simeq v_\infty t + (\Gamma t)^{1/3} \chi(\zeta, t)$

Measure statistics of χ

Tracy-Widom distribution of height

K. A. Takeuchi, M. Sano
(J. Stat. Phys. 2012)

Evidence for Geometry-Dependent Universal Fluctuations of the Kardar-Parisi-Zhang Interfaces in Liquid-Crystal Turbulence



Create interface by laser excitation of the darker phase either

- in a spot → circular surface
- or along a line → flat surface

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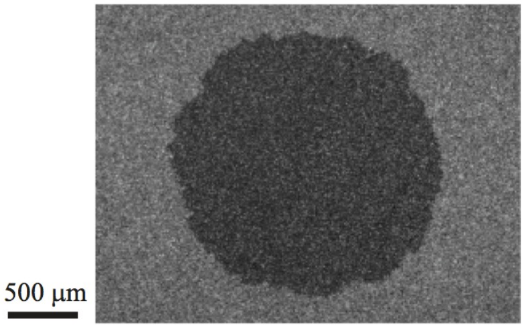
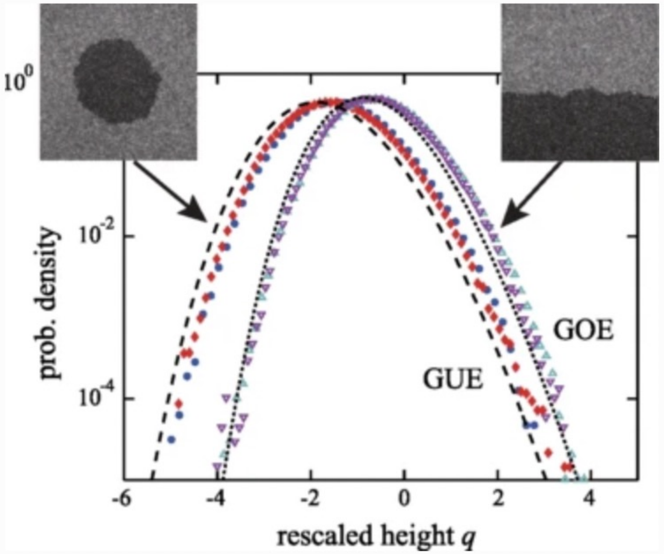


Fig. 8



Histogram of the rescaled local height $q\equiv(h-v_{\infty}t)/(\Gamma t)^{1/3}$ for the circular (*solid symbols*) and flat (*open symbols*) interfaces. The *blue circles* and *red diamonds* display the histograms for the circular interfaces at t=10 s and 30 s, respectively, while the *turquoise up-triangles* and *purple down-triangles* are for the flat interfaces at t=20 s and 60 s, respectively. The *dashed* and *dotted curves* show the GUE and GOE TW distributions, respectively, defined by the random variables χ_{GUE} and χ_{GOE}. (Color figure online)

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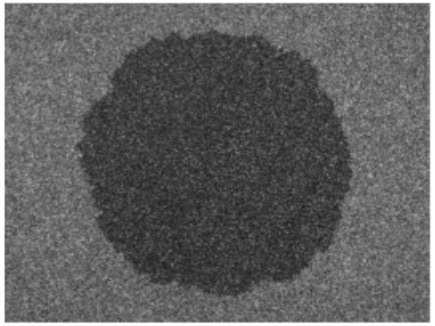

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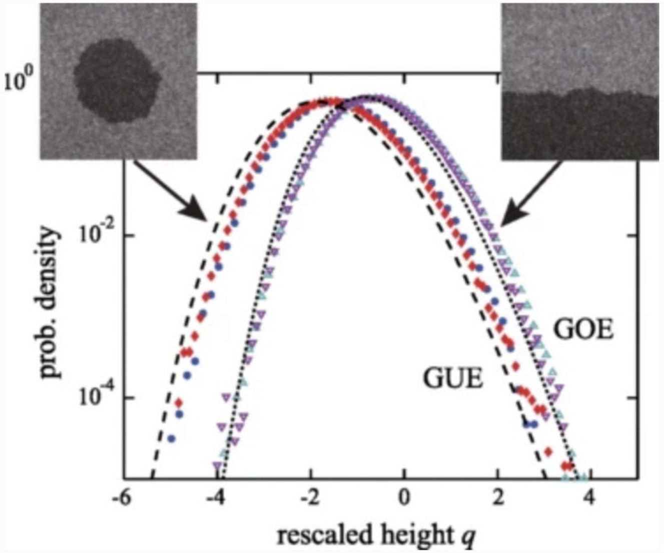

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Mirror symmetry of flat surface
↔
Time reversal symmetry of random matrix GOE

The directed polymer

A toy glass problem!

An instructive example for the crux of random partition functions

Very interesting fractal-like structure of configuration space:

Minima of polymers from apex to a chosen base line point

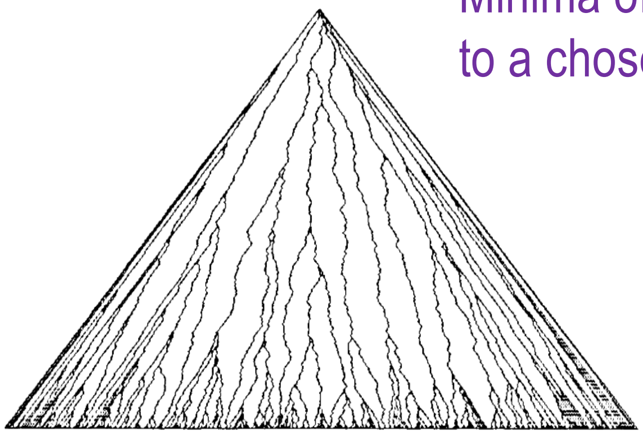


FIG. 2. A collection of polymers of lowest energy directed along the diagonals of a square lattice with random bonds. Each polymer (crossing 500 bonds) has one end fixed to the apex of the triangle, the other to various points on its base, and finds the optimal path in between.

Interesting questions?

- How does this picture change as we
  - change disorder
  - apply some force
- How hard is it to pull the polymer along the base line?

Sums over paths

appear in many circumstances:

- directed polymer (elastic line in disorder potential)
  As model for:
  - Domain wall between two ferromagnetic domains
  - Vortices in supercnductors

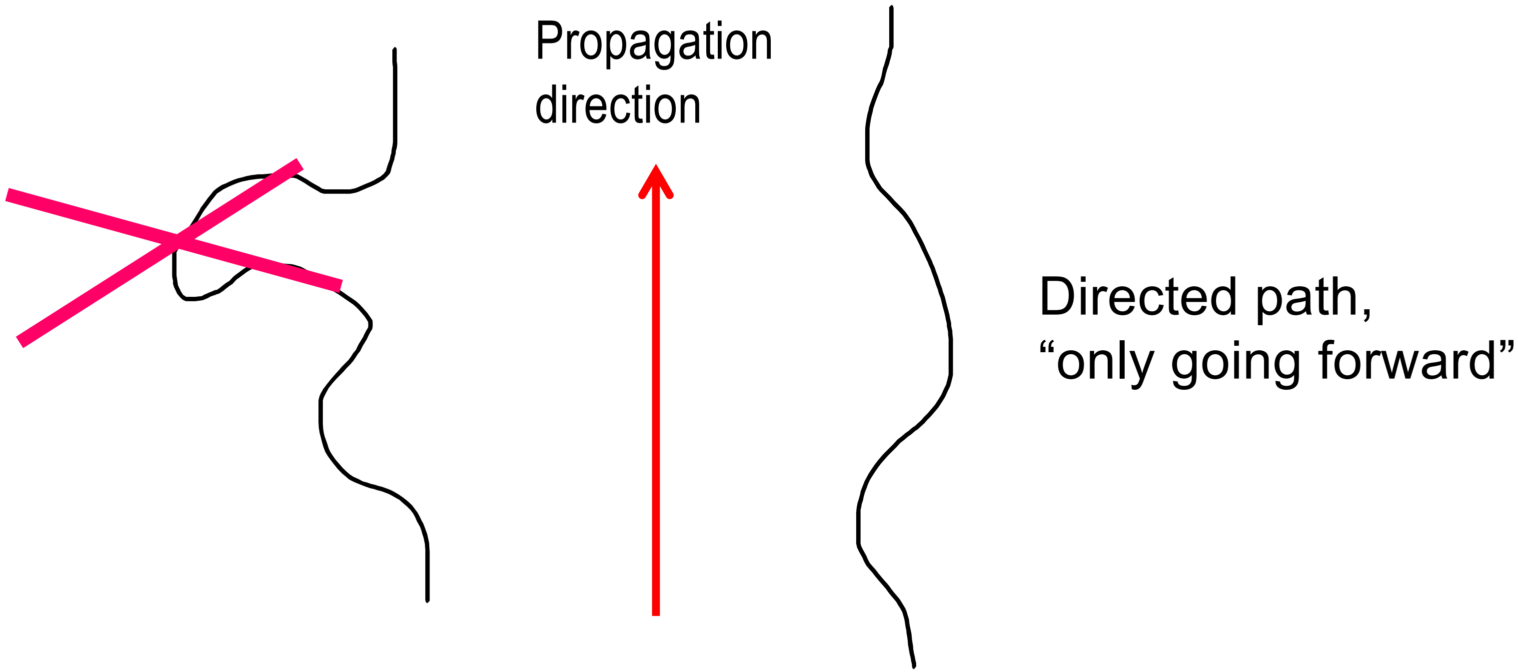
- High T expansion, e.g. of a random ferromagnetic Ising model:
  spin-spin correlator = sum over bonds connecting two points

- Decay of strongly localized quantum wavefunctions
  (path amplitudes can be negative/complex)

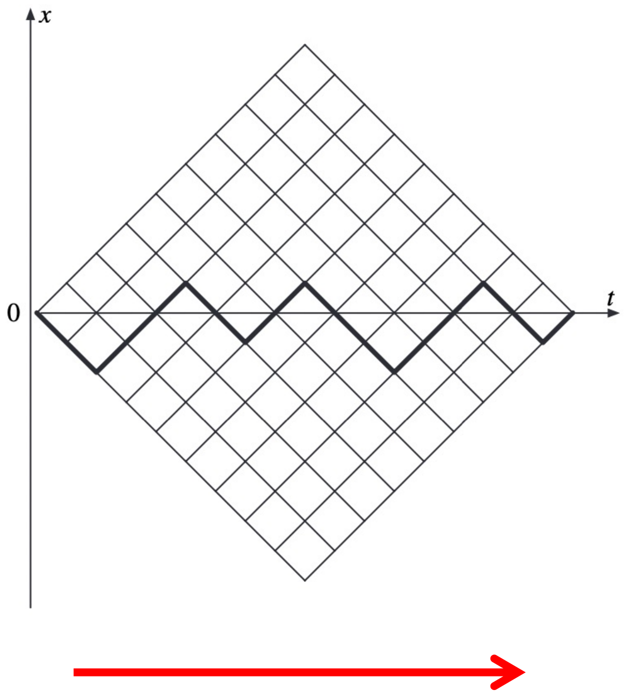
- Light propagation through heterogeneous (turbulent) medium (unitary)

Directed paths

Often, one can neglect overhangs of interfaces, at least at a coarse-grained level



Directed paths

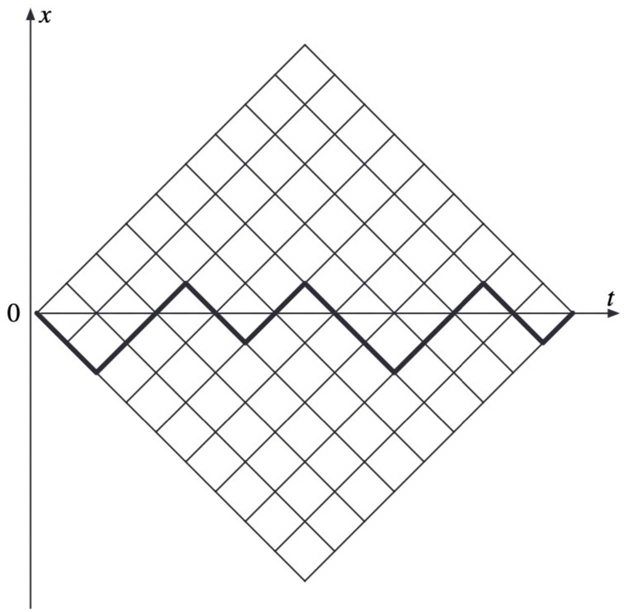


e.g. directed polymer on a lattice:

$$Z(X,t) = \sum_{\pi:\, x(0)=0 \to x(t)=X} e^{-\beta \sum_{0<t'\leq t} V(x(t'))}$$

(no elastic cost since all paths are equally long)

Directed paths



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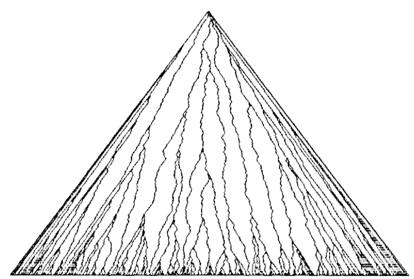
Convenient for numerical study:
Simple, fast recursion relation

$$Z(X,t) = e^{-\beta V(X)} \left[Z(X-1,t-1) + Z(X+1,t-1) \right]$$
$$Z(X,0) = \delta_{X,0}$$



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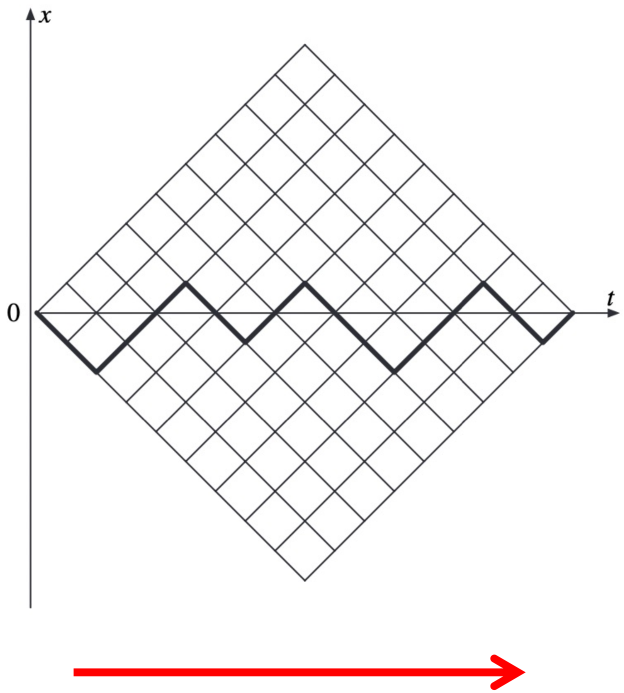

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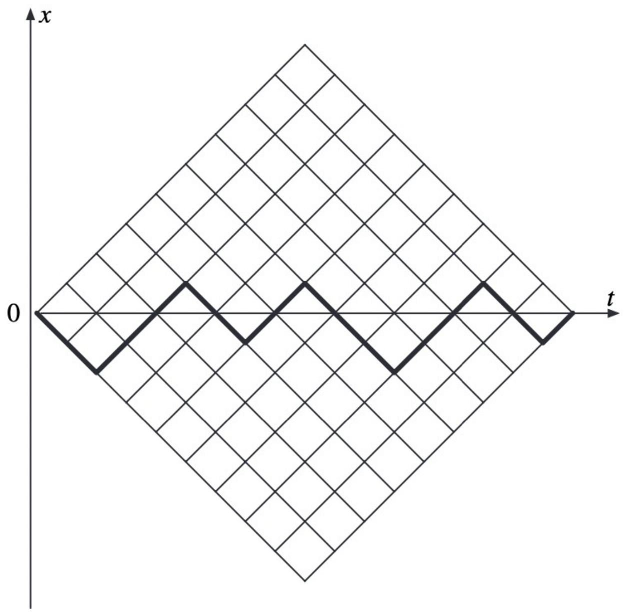
T = 0 limit looks like in TASEP:

$$E_{min}(X,t) = V(X) + \min \left[E_{min}(X-1,t-1), E_{min}(X+1,t-1) \right]$$

Directed paths



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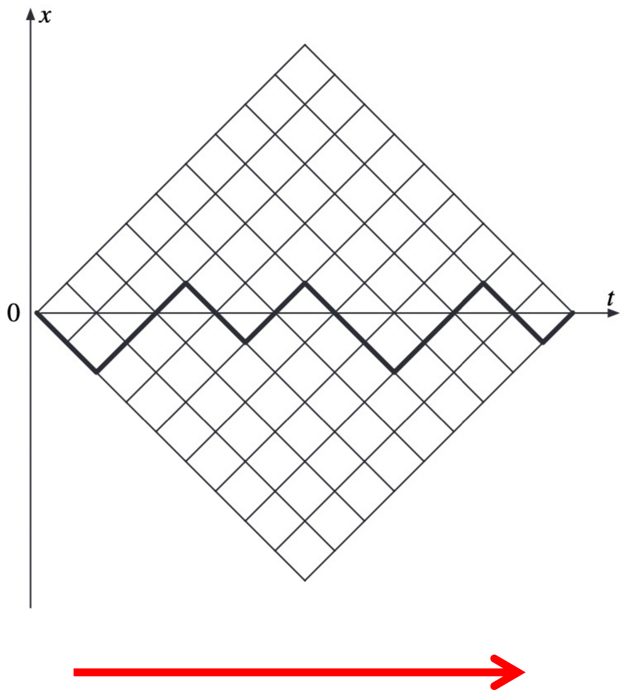
No disorder: simple 1d random walk!

Directed paths



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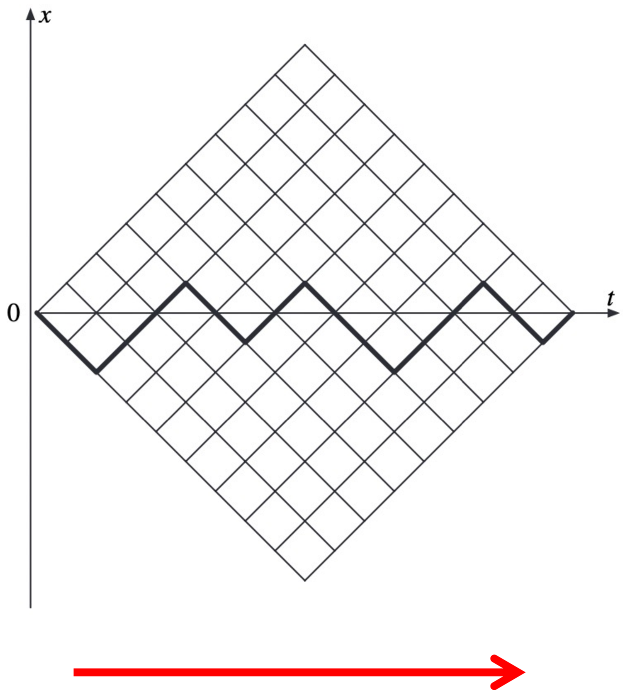
With disorder: Solve for Z in only time $O(t^2)$: not exponential in t!
$\longrightarrow$ Computationally easy, not NP–hard $\longrightarrow$ yet: properties of glasses arise!

Directed paths



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What can we do analytically?

Sums over paths: effect of disorder

Simplest case: 1d path of length *N* – a single term in the sum, with independent V_i.

The simple answer illustrates the difficulties arising from disorder:

$$Z = \prod_{i=1}^{N} e^{-\beta V_i} \qquad F = -\frac{\ln(Z)}{\beta} = \sum_{i=1}^{N} V_i$$

What can we say about the distribution of Z, or the free energy p(F)?

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What can we say about the distribution of Z, or the free energy p(F)?

$$p(Z) = \overline{\delta(Z - Z(\{V\}))}, \qquad \overline{(...)} = \text{disorder average}$$

The replica trick

Idea: compute $\overline{Z^n}$ for $n \in \mathbf{N}$ n copies or “replica”

Why? Integer powers of Z can be averaged easily and give a *non-disordered* system – but now of n copies.

$$Z = \prod_{i=1}^{N} e^{-\beta V_i} \qquad F = -\frac{\ln(Z)}{\beta} = \sum_{i=1}^{N} V_i$$

$$\overline{Z} = \overline{\prod_{i=1}^{N} e^{-\beta V_i}} = \overline{e^{-\beta V_i}}^{N}$$

independence of V_i

m’th cumulant

…

$$\overline{Z^n} = \overline{\prod_{i=1}^{N} e^{-n\beta V_i}} = \overline{e^{-n\beta V_i}}^{N} = \exp\left[N \sum_{m=1}^{\infty} \frac{n^m}{m!}(-\beta)^m \langle V^m \rangle_c\right]$$

Cumulants

Connected correlators or cumulants $\langle x^n \rangle_c$ of a random variable:

$$\overline{e^{-ikx}} \equiv \int e^{-ikx} p(x) dx =: \exp\left[\sum_{n\geq 0} \frac{(-ik)^n}{n!} \langle x^n \rangle_c\right]$$

$$\begin{aligned}
\langle x \rangle &= \langle x \rangle_c \\
\langle x^2 \rangle &= \langle x^2 \rangle_c + \langle x \rangle_c^2 \\
\langle x^3 \rangle &= \langle x^3 \rangle_c + 3\langle x^2 \rangle_c \langle x \rangle_c + \langle x \rangle_c^3
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The full correlator is the sum over partitions
of the product of connected correlators

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$$\begin{aligned}
\langle x \rangle_c &= \langle x \rangle \\
\langle x^2 \rangle_c &= \langle x^2 \rangle - \langle x \rangle^2 \\
\langle x^3 \rangle_c &= \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3
\end{aligned}$$

The full correlator is the sum over partitions of the product of connected correlators

Connected correlator = full correlator minus all connected components

Cumulants - Gaussians

Important case: Gaussian random variables

$$p(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

$$\overline{e^{-ikx}} = \int e^{-ikx} p(x) dx = \int e^{-ikx} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx = e^{-ik\mu - \frac{k^2\sigma^2}{2}}$$

$$\longrightarrow \quad \begin{aligned} \langle x \rangle &= \mu \\ \langle x^2 \rangle_c &= \sigma^2 \\ \langle x^{n>2} \rangle_c &= 0 \end{aligned}$$

Gaussians have only first and second cumulants!

Cumulants - Gaussians

Important case: Gaussian random variables – multidimensional Gaussian

$$\overline{e^{-ik^\nu x_\nu}} = \int e^{-ik^\nu x_\nu} p(x) d^d x = e^{-ik^\nu \langle x_\nu \rangle - \frac{k^\nu k^\mu \langle x_\nu x_\mu \rangle_c}{2}}$$

(Einstein convention: sum over indices)

Only cumulants are: means $\langle x_n \rangle$
and Gaussian covariance matrix
$\langle x_n x_m \rangle_c$

The replica trick

$$Z = \prod_{i=1}^{N} e^{-\beta V_i} \qquad F = -\frac{\ln(Z)}{\beta} = \sum_{i=1}^{N} V_i$$

$$\overline{Z^n} = \overline{\prod_{i=1}^{N} e^{-n\beta V_i}} = \overline{e^{-n\beta V_i}}^{N} = \exp\left[N \sum_{m=1}^{\infty} \frac{n^m}{m!}(-\beta)^m \langle V^m \rangle_c\right]$$

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If *F* were simply Gaussian, higher than second cumulants (m>2) would be absent!

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$\leftrightarrow$ Central limit theorem for *F* ?? (*F* Gaussian $\leftrightarrow$ *Z* log-normal*)*

If *F* were simply Gaussian, higher than second cumulants (m>2) would be absent!

- Indeed: good approximation for small moments *n!* (senses bulk of *P(Z)*)
- But: large moments Z^n are dominated by (non-universal) tails of *P(Z)*
  ($\leftrightarrow$ rare disorder configurations!)
- These tails and the too rapid growth (faster than n!) of high moments Z^n make the inference of P(Z) from its integer moments impossible (solution is not unique)

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$\leftrightarrow$ Up to which *n* is the “log normal” approximation good?

$$\overline{Z^n} \approx \exp\left[N \left(-n\beta\langle V \rangle + \frac{n^2}{2}\beta^2 \langle V^2 \rangle_c + \ldots \right) \right]$$

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$$n \ll n^* \sim T\frac{\langle V^2\rangle_c}{\langle V^3\rangle_c}$$

Higher cumulants can be neglected.

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BUT: always breaks down, even for n=1, for $T \lesssim T_* = \frac{\langle V^3\rangle_c}{\langle V^2\rangle_c}$

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Need a way to extrapolate moments to small n $\rightarrow$ 0 !
Ideally: Disorder-average the (*typical)* free energy,
not the partition function, which is dominated by rare disorder!

$$\boxed{\overline{F} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}}$$

Self-averaging

Note:

$$\overline{F} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$$

In contrast to Z, the free energy F is expected to be *self-averaging:*

For a large system, F is essentially a sum of many mutually independent subvolumes, each of which contains its own disorder realization. The total free energy of a thermodynamically large system ($N, V \to \infty$) thus automatically averages over disorder realizations, and one expects

$$F/N \overset{N\to\infty}{\longrightarrow} \overline{F}/N$$

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Difficulty: $\bar{F}$ is hard to compute. - But often one already gains insight by studying the integer cumulants of Z and try to extrapolate to small n. - Now apply to the DPRM!

Directed polymers: continuum

$$Z(X,t) = \int dx_0 Z(x_0,0) \int_{x_0(0)=x_0}^{x(t)=x} \mathcal{D}x(\tau) \exp\left[-\frac{1}{2}\int_0^t d\tau \left(\frac{dx}{d\tau}\right)^2 - \int_0^t d\tau V(x(\tau),\tau) \right]$$

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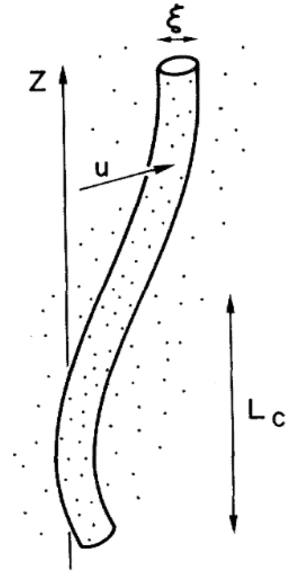
with Gaussian disorder: $$\overline{V(x,t)} = 0$$

Covariance: $$\overline{V(x,t)V(x',t')} = K(x-x')\delta(t-t')$$

Spatial correlations are usually of finite range ξ
due to the physical extent (thickness) of the line



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with Gaussian disorder: $$\overline{V(x,t)} = 0$$

Covariance: $$\overline{V(x,t)V(x',t')} = K(x-x')\delta(t-t')$$

Study fluctuations by looking at

$$\overline{Z(X_1,t)Z(X_2,t)...Z(X_n,t)} =$$

$$\int \prod_{m=1}^{n} dx_{m0} Z(x_{m0},0) \int_{x_m(0)=x_{m0}}^{x_m(t)=X_m} \prod_{m=1}^{n} \mathcal{D}x_m(\tau)$$

$$\exp\left[-\frac{1}{2}\sum_m \int_0^t d\tau \left(\frac{dx_m}{d\tau}\right)^2 + \frac{1}{2}\int_0^t d\tau d\tau' \sum_{m,m'=1}^{n} \overline{V(x_m(\tau),\tau)V(x_{m'}(\tau'),\tau')}\right]$$

$$= \int \prod_{m=1}^{n} dx_{m0} Z(x_{m0},0) \int_{x_m(0)=x_{m0}}^{x_m(t)=X_m} \prod_{m=1}^{n} \mathcal{D}x_m(\tau)$$

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Directed polymers: continuum

Back to a Schrödinger-like equation:

$$\psi(X_1, ..., X_n, t) := \overline{Z(X_1, t)Z(X_2, t)...Z(X_n, t)}$$

$$\partial_t \psi = -\mathcal{H}\psi$$

$$\mathcal{H} = -\frac{1}{2}\sum_{m=1}^{n} \partial^2_{X_m} - \frac{1}{2}\sum_{m,m'=1}^{n} K(X_m - X_{m'})$$

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Irrelevant constant energy per polymer

$$= \frac{1}{2}\sum_{m=1}^{n} \partial_{X_m}^2 - \frac{1}{2}\sum_{m\neq m'=1}^{n} K(X_m - X_{m'}) - \frac{n}{2}K(0)$$

Attractive two-body potential!

Directed polymers: continuum

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Attractive two-body potential!

Disorder has disappeared. BUT: a given polymer still feels where the low disorder potentials were: It is where one preferentially finds other polymer copies!
The disorder average indeed produces an attraction between replica.

Directed polymers: continuum

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“Solution”:

$$\psi(X_1, ..., X_n, 0) = Z(X_1, 0)Z(X_2, 0)...Z(X_n, 0) := \psi_0$$

$$\psi(X_1, ..., X_n, t) = \langle \mathbf{X}|e^{-\mathcal{H}t}|\psi_0\rangle = \sum_k \langle \mathbf{X}|\psi_k\rangle\langle\psi_k|e^{-E_k(n)t}|\psi_0\rangle$$

Directed polymers: continuum

Back to a Schrödinger-like equation:

$$\psi(X_1,...,X_n,t) := \overline{Z(X_1,t)Z(X_2,t)...Z(X_n,t)}$$

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$$\psi(X_1,...,X_n,t) = \langle\mathbf{X}|e^{-\mathcal{H}t}|\psi_0\rangle \overset{t\to\infty}{\approx} \langle\mathbf{X}|\psi_{\mathrm{GS}}\rangle\langle\psi_{\mathrm{GS}}|e^{-E_{\mathrm{GS}}(n)t}|\psi_0\rangle$$

Directed polymers: continuum

Long propagation: Ground state of the n-polymer problem?

$$\mathcal{H}' = \frac{1}{2}\sum_{m=1}^{n} \partial^2_{X_m} - \frac{1}{2}\sum_{m \neq m'=1}^{n} K(X_m - X_{m'})$$

ψ_{GS} ?

Consider short range correlations, $K(x) \to g\delta(x)$

$\longrightarrow$ Lieb-Liniger repulsive 1d Bose gas (as probed in cold atoms!).

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- Ground state: single bound state of all n particles (Bethe ansatz string) with wavefunction

$$\psi_G(X_1, ..., X_n) = \exp\left[-\frac{g}{4}\sum_{m\neq m'} |X_m - X_{m'}|\right]$$

$$\boxed{E_G = -\frac{g^2}{24}(n^3 - n)}$$

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Scaling analysis of a bound state of n particles, of size R :

$$E_{\mathrm{bd}}(n,R) \sim \frac{n}{R^2} - g\frac{n^2}{R}$$

$$R_{\mathrm{min}} \sim \frac{1}{gn}$$

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- Rationalizes the n^3 scaling
- Breakdown of delta function approximation for n > n* ~ $1/g\xi$ $\longleftrightarrow$ $R_{\mathrm{min}}(n^*) \sim \xi$
- Beyond: non-universal behavior depending on extreme values of local potential

Directed polymers: continuum

How to infer something about $p(Z)$ from the moments at $n < n^*$?

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$$\overline{Z^n(X,t)} \propto \exp(Cnt + \frac{g^2}{24}t(n^3 - n)) \stackrel{?}{=} \exp\left[\sum_k \frac{n^k}{k!}\overline{(-\beta F)^k}^c\right]$$

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$t \to \infty$ at fixed n $\longleftrightarrow$ Expansion around $n \to 0$, at fixed t

These limits do not commute!

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- Replica approach yields independently the KPZ/DPRM energy exponent $\boxed{\theta = \frac{1}{3}}$!

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- Beyond the above, the replica approach allows to analyze finite range correlators, T effects etc.

Exercises

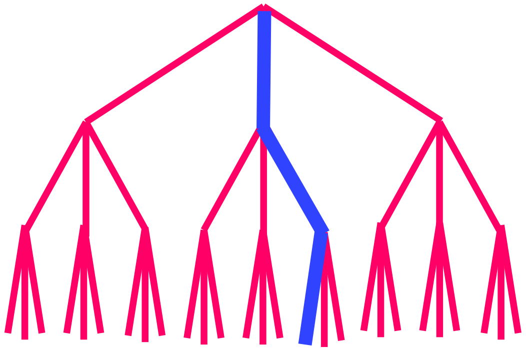
Replica approach to DPRM

- Check the ground state wavefunction of the n-polymer problem and compute its energy.
- Consider D=2 transverse directions instead of 1. Do you expect a bound state? Repeat the scaling analysis for this case. Does it allow to conclude anything about the scaling of energy with n and thus for the energy exponent θ?

Exercises

Directed polymer on a Cayley tree

- Consider a Cayley tree with branching number k



k=3

- A polymer goes from any leaf (bottom to the top), its energy being the sum of site energies E_i encountered along the path, $E = \sum_{i \in path} E_i$
- Derive a recursion equation for the partition function (at finite T), as you progress from the leaves towards the root.
- Derive a recursion for the zero temperature limit (minimal energy configuration)!

Exercises

- At generation t from the leaves define the partition function Z(t) and the quantity

$$G_t(x) = \overline{\exp(-e^{-\beta x} Z(t))}$$

- Show that the above recursion implies that

$$G_{t+1}(x) = \int dV \rho(V) [G_t(x+V)]^k$$

This can be shown to have travelling wave solution of the form w(x -ct). Determining c amounts to computing the free energy density.